Theoretical Computer Science 411 (2010) 285–287

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

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A comprehensive analysis of degree based condition for Hamiltonian cycles

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ARTICLE INFO

Article history:
Received 27 October 2008
Received in revised form 2 September 2009
Accepted 13 September 2009
Communicated by D. Peleg

Keywords:
Hamiltonian cycle
Hamiltonian path
Graphs

ABSTRACT

Since finding whether a graph has a Hamiltonian path or Hamiltonian cycle are both NP-complete problems, researchers have been formulating sufficient conditions that ensure the path or cycle. Rahman and Kaykobad (2005) [2] presented a sufficient condition for determining the existence of Hamiltonian path. Three recent works – Lenin Mehedy, Md. Kamrul Hasan, Mohammad Kaykobad (2007) [3], Rao Li (2006) [4], Shengjia Li, Ruijuan Li, Jinfeng Feng (2007) [5] – further used the same or similar condition to ensure Hamiltonian cycle with some exceptions. The three works, along with their unique findings, have some common results. This paper unifies the results and brings them under Rahman and Kaykobad's condition.

1. Preliminaries

We consider only simple undirected graphs- graphs that do not contain loops or multiple edges. A Hamiltonian cycle is a closed path passing through every vertex of a graph. A graph containing a Hamiltonian cycle is said to be Hamiltonian. By Hamiltonicity we mean the virtue of a graph to be Hamiltonian. Naturally every Hamiltonian graph contains a Hamiltonian path but a graph with Hamiltonian path may not contain a Hamiltonian cycle. We define $\delta(u, v)$ as the shortest distance between u and v. Let us take the notation $\text{diam}(G)$ as the diameter of G, K_p as the complete graph with p vertices, $K_{p,q}$ as complete bipartite graph with $p + q$ vertices. A complete graph is also called a clique. We term the condition: "$d(u) + d(v) + \delta(u, v) \geq n + 1$, *where* $|V| = n$ *and* u, v are distinct non-adjacent vertices of G", as Rahman–Kaykobad condition.

We also need to describe two graph families C_n, D_n. We derive two special classes of graph from C_n, namely C'_n and C''_n.

C_n consists of graphs formed from two cliques A and B and a vertex, w connected to at least one vertex in A and at least one vertex in B. Note that the maximum diameter of C_n is 4, when w is not connected to at least one vertex of clique A and at least one vertex of clique B.

We define C'_n as a special case of C_n when w is connected to all the vertices of clique A. When w is connected to all the vertices of both cliques, A and B, C_n becomes C''_n. Here the diameters of C'_n and C''_n are 3 and 2 respectively.

D_n is defined as $\{G : K_{p,p+1} \subseteq G \subseteq K_p + (p+1)K_1$, where $p \geq 1$, and $(p+1)K_1$ denotes $(p+1)$ isolated vertices, and $+$ is the join operator$\}$. D_n is a 2-connected graph and has a diameter of 2.

The graph classes C_n, D_n are presented visually in Figs. 1 and 2. C_n was defined in [4] and D_n was defined in [4] and also in [5] as L_n. Here, for the sake of simplicity, we use slightly different but equivalent definitions. It is to be noted that graphs in $C_n \cup D_n$ are not Hamiltonian but contain a Hamilton Path.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2009.09.018

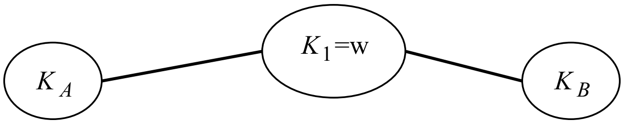

Fig. 1. Graph class C_n.

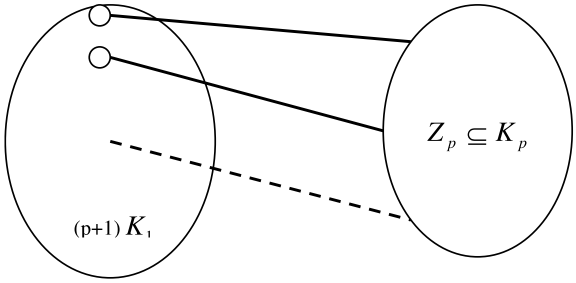

Fig. 2. Graph class D_n.

2. Existing results

Now we discuss some earlier results.

Theorem 2.1 (*Ore [1]*). *If $d(u) + d(v) \geq n$ for every pair of distinct non-adjacent vertices u and v of G, then G is Hamiltonian.*

Theorem 2.2 (*Rahman and Kaykobad [2]*). *Let G be a connected graph which satisfies Rahman–Kaykobad condition, then G has a Hamiltonian path.*

Lemma 2.1 (*Mehedy, Hasan and Kaykobad [3]*). *Let G be a 2-connected graph which satisfies the Rahman–Kaykobad condition. Then the endpoints of any Hamiltonian path are at distance at most 3.*

Theorem 2.3 (*Mehedy, Hasan and Kaykobad [3]*). *Let G be a 2-connected graph which satisfies the Rahman–Kaykobad condition. If G contains a Hamiltonian path with endpoints at distance 3 then G is Hamiltonian.*

Theorem 2.4 (*Rao Li [4]*). *Let G be a connected graph which satisfies the Rahman–Kaykobad condition, then G is Hamiltonian or $G \in C_n \cup D_n$.*

Theorem 2.5 (*Li, Li and Feng [5]*). *Let G be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of vertices u and v with $\delta(u, v) = 2$, then G is Hamiltonian, unless n is odd and $G \in D_n$.*

Ore [1] in Theorem 2.1, proposed a sufficient condition for a graph to be Hamiltonian. A graph with Ore's condition has a diameter of only 2. But if a sufficient condition can be derived for a graph with diameter more than 2, Hamilton Path or cycle may be found with fewer edges. With this motivation, Rahman and Kaykobad [2] proposed a sufficient condition to find Hamilton Path in a graph as given in Theorem 2.2.

Mehedy, Hasan and Kaykobad [3] extended the result of Rahman and Kaykobad [2] that Rahman–Kaykobad condition for a 2-connected graph ensures Hamiltonian Cycle when any existing Hamilton Path's end points are at a distance of 3, as stated in Theorem 2.3. From Lemma 2.1, it can be concluded that the diameter of the graph will be maximum 3.

Rao Li [4] followed Rahman–Kaykobad condition but did not consider 2-connectedness. He showed that, if the graph is not Hamiltonian, it falls into any of the two classes of graphs. He did not discuss anything about the diameter of the graph.

Li, Li and Feng [5] worked on Ore's condition and further restricted the condition for a 2-connected graph. They found that, with the reduced number of edges, sometimes the graph may not be Hamiltonian. They characterized the non-Hamiltonian graphs as a special graph class, also described by Rao Li [4].

In this paper, we propose a unified theorem for Rahman–Kaykobad condition based on existing results. We find out what is the maximum diameter of a graph under Rahman–Kaykobad condition. Then we meticulously describe whether the graph is Hamiltonian, and if not, what graph class it will fall into.

3. Main results

Theorem 3.1. *Let $G = (V, E)$ be a connected graph with n vertices such that for all pairs of distinct non-adjacent vertices $u, v \in V$ we have $d(u) + d(v) + \delta(u, v) \geq n + 1$ (i.e. G satisfying Rahman–Kaykobad condition), then*

(i) $\mathrm{diam}(G) \leq 4$;
(ii) *if* $\mathrm{diam}(G) = 4$, *then* $G \in C_n$;
(iii) *if* $\mathrm{diam}(G) = 3$ *then G is Hamiltonian if* 2-*connected or else* $G \in C'_n$;
(iv) *if* $\mathrm{diam}(G) = 2$:

– *If G is 2-connected, then G is Hamiltonian when n is even and G is in D_n when n is odd.*
– *Otherwise G is in C''_n.*

(v) *Otherwise, G is a clique.*

Proof.

Proof of (i) and (ii). If a graph $G = (V, E)$ fulfills Rahman and Kaykobad condition, Theorem 2.4 [4] says that G is Hamiltonian or $G \in C_n \cup D_n$. If G is Hamiltonian, it is 2-connected by definition. So, by Lemma 2.1 [3], $\mathrm{diam}(G) \leq 3$, since any two consecutive vertices on a Hamiltonian cycle are endpoints of a Hamiltonian path. But if G is not Hamiltonian, G falls into graph class C_n *with* $\mathrm{diam}(G) \leq 4$ or G *is in* D_n *with* $\mathrm{diam}(G) = 2$. This immediately implies (i) and (ii).

Proof of (iii). When $\mathrm{diam}(G) = 3$, the graph is Hamiltonian or of class C_n. If the graph is 2-connected, it cannot be an instance of C_n, so it is Hamiltonian. Now suppose G has cut vertex, and so G is of class C_n. Removing the cut vertex w from G, we get two cliques A and B. As $\mathrm{diam}(G) = 3$, w cannot be connected to both A and B, assume without loss of generality, the former. Consequently, at least one vertex of B cannot be connected to w. That proves G in C'_n.

Proof of (iv). It follows immediately from Theorem 2.4 [4] and Theorem 2.5 [5].

4. Conclusion

We relate three seminal works for finding the Hamiltonicity of a graph observing Rahman and Kaykobad condition or a subset of the condition. We determine when the graph is Hamiltonian. If we cannot, we characterize the non-Hamiltonian graph by using two families of graphs, and in which instances of graphs in those two families the graph falls. Our work provides a comprehensive understanding of graphs following Rahman and Kaykobad condition.

Acknowledgments

We would like to thank the anonymous reviewers for their valuable comments and suggestions that significantly helped to improve this paper.

This research was supported by the MKE (Ministry of Knowledge Economy), Korea, under the ITRC (Information Technology Research Center) support program supervised by the IITA (Institute of Information Technology Advancement)" (IITA-2009-(C1090-0902-0002)) and was supported by the IT R&D program of MKE/KEIT, [10032105, Development of Realistic Multiverse Game Engine Technology].

This work also was supported by the Brain Korea 21 projects and Korea Science & Engineering Foundation (KOSEF) grant funded by the Korea government (MOST) (No. 2008-1342).

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